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Solving Helmholtz Equation by Means of Domain Decomposition Using Discontinuous Galerkin and Pseudospectral Methods

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SUMMARY

In this work we introduce and study numerically a new method for solving two-dimensional Helmholtz equation in complex media with high-contrast velocity variations. The proposed method is based on coupling the IPDG method for solving the problem in the surface neighbourhood with fast pseudospectral method applied in the lower part of the domain. The coupling is implemented using overlapping domain decomposition with Robin transmission conditions. Results of numerical experiments with both analytical solutions and point-like sources are presented for Gullfaks benchmark.

Introduction

Development of efficient algorithms for solving Helmholtz problem in domains of complicated shape still remains a challenging topic in numerical modelling although a lot of research has been done on the subject already. Helmholtz problem naturally arises, for instance, in wave propagation modelling in frequency domain, which is widely used for full waveform inversion and reverse time migration algorithms. On the one hand, there exists a strong theoretical background for the Helmholtz problem concerning, in particular, well-posedness and unique solvability for a variety of formulations in a large class of domains. On the other hand, after discretization of the problem one obtains a large system of linear algebraic equations (SLAE) with sparse indefinite symmetric, but non-Hermitian matrix. Polygonal domains with artificial boundary conditions, variable coefficients and oscillating behaviour of fundamental solutions also require a very careful treatment from computational point of view.

Traditionally, for discretization of the Helmholtz equation finite difference (FDM) or finite element methods (FEM) are applied. However, these methods have several drawbacks. In most applications in geoseismic modeling the upper boundary of the domain is assumed to be a polygonal curve (Earth surface). In addition, high-contrast media implies that the velocity field usually consists of several layers corresponding to the inner geometry of the domain at hand. In order to resolve such specific features of the inner geometry one has to use very fine meshes at least in the upper part of the domain. Finite difference schemes were extensively used in the past for solving Helmholtz equation. However, they don't fit very well into the framework of unstructured meshes and often require very small discretization step for describing the high-contrast media accurately enough. Recently, a new type of numerical algorithms has gained great attention, so called discontinuous Galerkin methods (DG), see, e.g. Cockburn et.al. (2000). Unlike FDM they can handle complicated geometrical structures and unlike FEM they don't suffer from the "pollution" effect. As for Helmholtz equation, interior penalty discontinuous Galerkin method (IPDG), see Feng and Wu (2009), proved to be a very efficient numerical method. However, for DG method the resulting SLAE is usually much larger than for FDM of the same order. Moreover, triangulating the entire domain appears to be very costly in practice.

Another approach to solving Helmholtz equation is pseudospectral methods, as in Neklyudov et.al. (2014). The idea of this method is to precondition the Helmholtz operator of the initial problem by the Helmholtz operator in a simple domain with layered velocity field which allows one to use very fast methods to compute the preconditioner action. Obviously, this preconditioner would work only in case of rectangular domains covered with uniform structured mesh and behaves not well when the media is high-contrast. However, the method still seems attractive since the computational costs are much lower than for DG method.

In this work we couple the IPDG method for solving the Helmholtz equation in the upper part of the domain with high-contrast media and pseudospectral method in the lower part of the domain. The idea is based on a simple remark that usually the media is extremely high-contrast in the upper part of the domain but in the lower part of the domain velocity field does not change so drastically. The coupling is implemented by overlapping domain decomposition (DD) with Robin-type transmission conditions at the interfaces. The resulting method enjoys both high accuracy of DG near surface and computationally efficient solution by pseudospectral algorithm in the lower part of the domain.

Problem statement

As the initial problem we consider the Helmholtz equation $-\Delta \boldsymbol{u} - \boldsymbol{\omega}^2 \boldsymbol{V}^{-2} \boldsymbol{u} = \boldsymbol{f}$ for $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\Omega}$, where $\boldsymbol{\Omega} \subset \mathbb{R}^2$ is a bounded polygonal domain. Frequency $\boldsymbol{\omega}$ is constant; velocity field $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{y})$ can vary due to the inner structure of the domain rather intensively.

Domain decomposition

Domain decomposition is widely exploited for solving different PDE's in complex domains with variable coefficients. It is also a standard tool for coupling different numerical methods used in different (non-)overlapping parts of the domain. The most famous DD method is the alternating Schwarz algorithm which was initially proposed in 1980s for elliptic problems in works by Matsokin and Nepomnyashchikh (1985). Application of domain decomposition methods for solving Helmholtz equation can be found in Collino et. al. (1998), Gander et. al. (2007).
Briefly, the idea of DD can be represented for the case of two overlapping domains, i.e. $\Omega = \Omega_1 + \Omega_2$ with interfaces Γ_1 and Γ_2, in the following way. The initial problem is replaced by two subdomain problems

$$\begin{cases} -\Delta u_1 - \frac{\omega^2}{V^2} u_1 = \boldsymbol{f}, (x,y) \in \Omega_1 \\ +boundary\ cnd \\ +transmission\ cnd\ at\ \Gamma_1 \end{cases} \quad \begin{cases} -\Delta u_2 - \frac{\omega^2}{V^2} u_2 = \boldsymbol{f}, (x,y) \in \Omega_2 \\ +boundary\ cnd \\ +transmission\ cnd\ at\ \Gamma_2 \end{cases},$$

where one can choose a suitable numerical method in each subdomain. Due to the indefiniteness of Helmholtz operator, transmission boundary conditions are of Robin type at Γ_1 and Γ_2 (with $\boldsymbol{n}$ as the corresponding exterior normal vector), i.e.: $\frac{\partial u_1}{\partial \boldsymbol{n}} + i\omega u_1 = \frac{\partial u_2}{\partial \boldsymbol{n}} + i\omega u_2$.

Discontinuous Galerkin method

The main difference between finite element and discontinuous Galerkin methods is the choice of basis functions which are assumed to be completely discontinuous across elements in DG. This implies the necessity of stabilization through penalty terms associated with jumps of the solution (and possibly, its derivatives) across interelement boundaries. In this work we use a variant of IPDG method studied in Feng and Wu (2009). It can be written in the following form (omitting integrals over domain boundary):

$$Find\ u_h \in U_h\ such\ that$$
$$a_h(u_h, v_h) - k^2(u_h, v_h) = (f, v_h), \ \ \forall v_h \in U_h$$

where $k^2 \in \mathbb{C}$ and bilinear form a_h is defined as

$$a_h(u,v) = \sum (\nabla u, \nabla v)_K - \sum \left(\left(\left\{ \frac{\partial u}{\partial n_e} \right\}, [v] \right)_e + \left([u], \left\{ \frac{\partial v}{\partial n_e} \right\} \right)_e \right) + \boldsymbol{i} J(u,v)$$

and penalty term $J(u,v)$ is associated with interelement jumps of the solution

$$J(u,v) = \sum \frac{\gamma_{0,e}}{h_e} \langle [u], [v] \rangle_e.$$

Here K denotes a triangulation element with edges e, $[\cdot]$ and $\{\cdot\}$ are usual jump and average operators at e, and, finally, $(\cdot,\cdot)_D$ is standard inner product in $L_2(D)$. The finite-dimensional space U_h consists of functions which are element-wise polynomials of certain degree.

Pseudospectral solver by Neklyudov et. al.

The initial Helmholtz problem can be written in the operator form

$$Lu \equiv \Delta u + \omega^2 V^{-2} u = f$$

The proposed right preconditioner with depth(z)-dependent coefficient L_0 is defined as

$$L_0 u = \Delta u + \omega^2 \big(1 + \beta(z)\big) V_0^{-2}(z) u.$$

After preconditioning and discretization the resulting SLAE is solved by the induced dimension reduction method (IDR). The preconditioner is then inverted by a fast method which combines two-dimensional Fourier transform in lateral directions with a finite difference method or semi-analytical solution of the ODE in z-direction with piecewise-constant coefficient. The depth-dependent velocity $V_0(z)$ is obtained by averaging velocity variation in lateral directions and piecewise-constant approximation in z on a fine mesh. The main advantage is that one can use a structured mesh with small number of nodes in lateral directions and, roughly speaking, reduce the problem at hand to one-dimensional (which significantly reduces size of the overall problem).

Numerical experiments

To study the performance and applicability of the presented approach a wavefield of the point source was computed for Gullfaks model. The three-level preconditionner was used. First, the Helmholtz problem was preconditioned by shifted Laplacian approach. The main idea then is to couple solution of the problem on coarse mesh in the simple domain by spectral methods with describing accurately the non-smooth behaviour of the solution in the near-surface region. To this end, the second preconditioner was based on overlapping two-domain decomposition with Robin transmission conditions, as described above. The equation in the upper domain with polygonal boundary and strong variations of velocity field (up to an order of magnitude) was solved by IPDG method whereas for lower rectangular domain a pseudospectral preconditioner was used. Numerical results are presented for analytical tests as well as real-life application problem based on Gullfaks benchmark velocity field, with point-like sources.

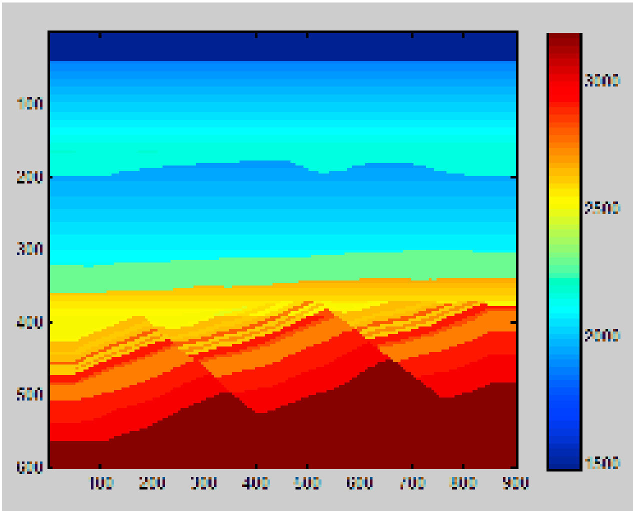

Figure 1 *A part of velocity field (M/s) in Gullfaks benchmark.*

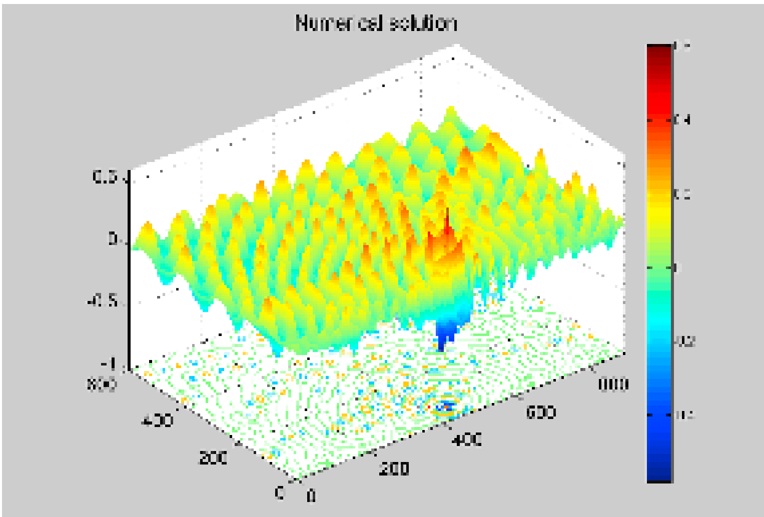

Figure 2 *A typical solution of Helmholtz problem (obtained by a discontinuous Galerkin method) in a homogeneous media (velocity v=700 M/s) for a point-like source (frequency w ~ 100). PML boundary conditions are used at x=0 and x = 900; ABC boundary conditions – at y = 0 and y=600. Mesh was generated by Triangle® software.*

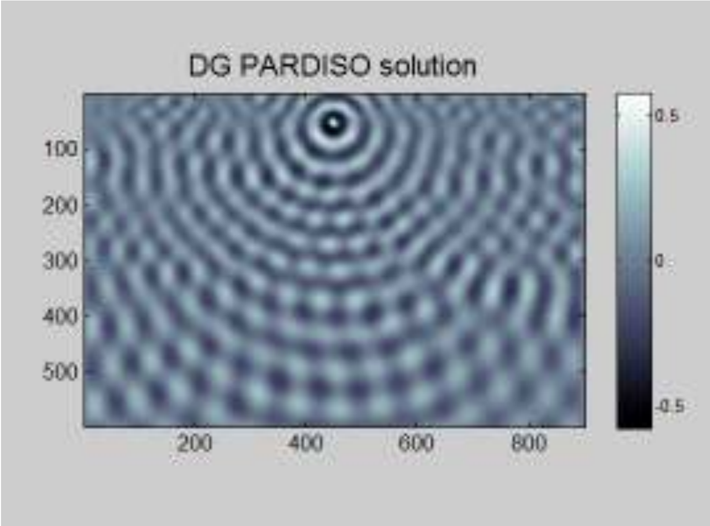

Figure 3 *Wavefield for the solution shown in Figure 2.*

Conclusions

In this work we introduce and study numerically a new method for solving Helmholtz equation in complex media with high-contrast velocity variations. The proposed method is based on coupling of the IPDG method for solving the problem near surface with fast pseudospectral method applied in the lower part of the domain. The coupling is implemented using overlapping domain decomposition with Robin transmission conditions. Results of numerical experiments with both analytical solutions and point-like sources are presented for Gullfaks benchmark.

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